Chemistry Solutions

1. 1 litre water = 1kg i.e. 1000 g water ($\because$ d = 1000 kg/m^3)

$$\equiv \frac{1000}{18} = 55.55 \text{ moles of water}$$

So, molarity of water = 55.55M

2. $C_{rms} = \sqrt{\frac{3RT}{M}}, C_{av} = \sqrt{\frac{8RT}{\pi M}}$

$$\frac{C_{rms}}{C_{av}} = \sqrt{\frac{3RT}{M}} \times \sqrt{\frac{\pi M}{8RT}} = \sqrt{\frac{3\pi}{8}} = 1.085$$

$C_{rms} = 1.085 \times C_{av}$

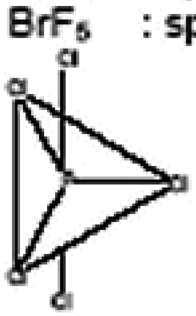
$= 1.085 \times 400$

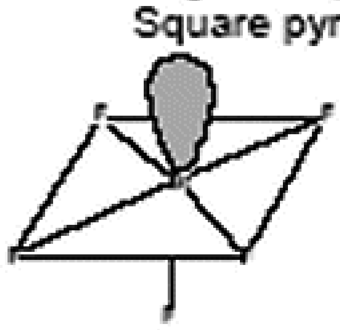
$= 434 \text{ ms}^{-1}$

3. Ziggler Natta catalyst ($R_3Al + TiCl_4$)

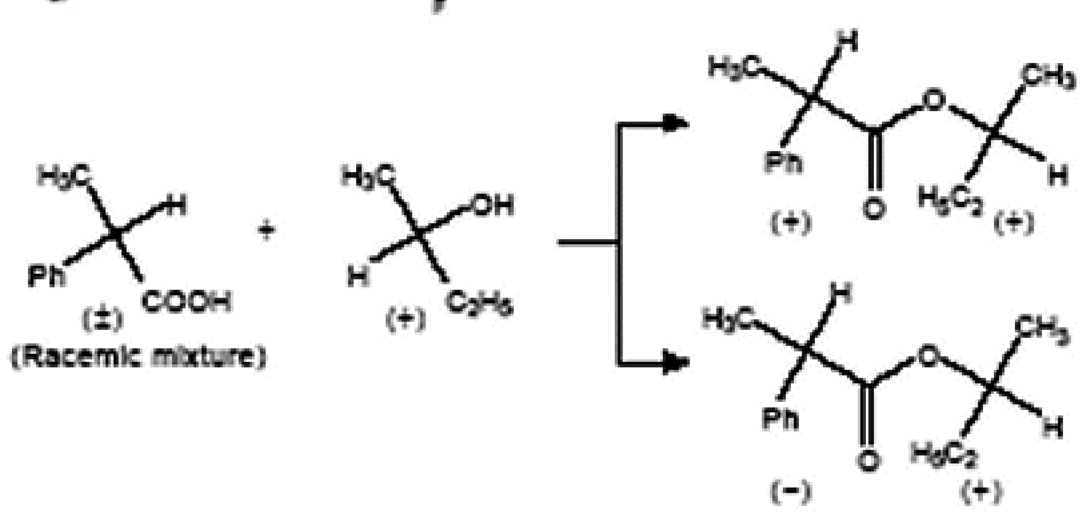
4. Oxygen atom of diethyl ether by donation of its lone pair to vacant 3p orbitals of Al in anhydrous $AlCl_3$ solvates it more compared to hydrous $AlCl_3$.

5. PCl_5 : sp^3d Trigonal bipyramid
BrF_5 : sp^3d^2 Square pyramidal





6.



The bonds attached to the chiral carbon in both the molecules are not broken during the esterification reaction. (+) Acid reacts with (+) alcohol to give an (++) ester while (−) acid reacts with (+) alcohol to give (+ −) ester. These two esters are diastereoisomers.

7. $$\frac{1}{\lambda} = R_H . Z^2 \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right)$$

R_H is a constant and transition remaining the same,

$$\frac{1}{\lambda} \propto Z^2$$

$$\frac{\lambda_{He}}{\lambda_H} = \frac{Z_H^2}{Z_{He}^2} = \frac{1}{4}$$

So, $\lambda_{He} = \frac{1}{4} \times 91.2 = 22.8$ nm

8.

		K_a value
a)	Benzoic acid	6.3×10^{-5}
b)	$p-NO_2-C_6H_4-COOH$	30.6×10^{-5}
c)	$p-Cl-C_6H_4-COOH$	6.4×10^{-5}
d)	$p-CH_3-C_6H_4-COOH$	4.3×10^{-5}
e)	$p-OCH_3-C_6H_4-COOH$	3.3×10^{-5}

9. $2PbS + 3O_2 \longrightarrow 2PbO + 2SO_2$

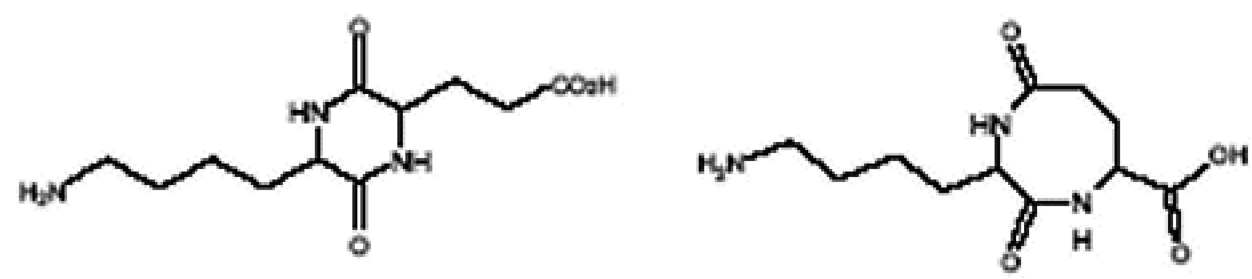
$PbS + 2O_2 \longrightarrow PbSO_4$

$PbS + 2PbO \longrightarrow 3Pb + SO_2$

$PbS + PbSO_4 \longrightarrow 2Pb + 2SO_2$

Oxidation number of Pb in litharge (PbO) is +2.

10.



11. a) Area of quadrilateral = $2\frac{1}{2} \times a \times a \times \sin\theta = a^2 \times \sin\theta$

Where a = length of the side of the quadrilateral

To have the maximum area, i.e. $\sin\theta = 1$

or $\theta = 90°$. In other words, the quadrilateral must be a square

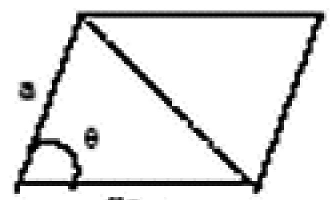


fig.1

Area of square = 4 × 4 = 16 cm^2

Again to have the maximum no. of spheres the packing must be hcp

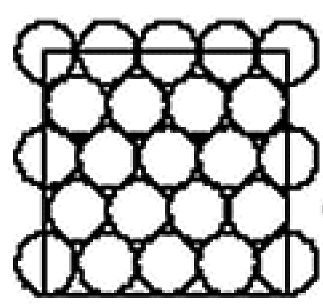


fig.2

Maximum no. of sphere s= 18 (see fig. 2)

Area = 16 sq. cm

$\therefore$ No. of spheres per $cm^2 = \frac{18}{16} = 1.125$

b) No. of m mole of CH_3COOH initially taken = 100 × 0.5 = 50

Since concentration reduces to 0.49 M

$\therefore$ Final no. of m mole of CH_3COOH = 100 × 0.49 = 49

$\therefore$ No. of m mole of CH_3COOH get adsorbed

= 50 – 49 = 1

$\therefore$ No. of molecules of CH_3COOH get adsorbed = 6.02×10^{20}

Since 1g charcoal has area = 3.01×10^2 m^2

$\because$ 6.02×10^{20} molecules of acetic acid gets adsorbed in 3.01×10^2 m^2 area

$\therefore$ 1 molecule of acetic acid gets adsorbed = $\frac{3.01 \times 10^2}{6.02 \times 10^{20}} = \frac{1}{2} \times 10^{-18} = 5 \times 10^{-19}$ m^2

12. a) At 25°C: $K_w = 10^{-14}$ $pK_w = 14$ $\therefore$ pH + pOH = 14

Pure water being neutral, pH = pOH = 7

As temperature decreases, K_w decreases and hence pK_w increases. Obviously, pH of water which is $pK_w/2$ will increase. Thus, pH of water at 4°C will be more than that at 25°C.

b) Daniel cell is: $Zn \mid Zn^{2+} \parallel Cu^{2+} \mid Cu$

Let there be two Daniel cells with their E_{cell} as given below:

$Zn \mid Zn^{2+} (C_1) \parallel Cu^{2+} (C = ?) \mid Cu,$ $\qquad E_{cell} = E_1$

$Zn \mid Zn^{2+} (C_2) \parallel Cu^{2+} (C = 0.5\ M) \mid Cu$ $\qquad E_{cell} = E_2$ where $E_2 > E_1$

From question

$E_2 - E_1 = 0.03$ and $C_2 = C_1$

The cell reaction is $Zn + Cu^{2+} \longrightarrow Zn^{2+} + Cu$, $Q = \frac{[Zn^{2+}]}{[Cu^{2+}]}$

So, $E_{cell} = E^0_{cell} - \frac{0.06}{2}\log\frac{[Zn^{2+}]}{[Cu^{2+}]}$

Thus, $E_1 = E^0_{cell} - \frac{0.06}{2}\log\frac{C_1}{C}$

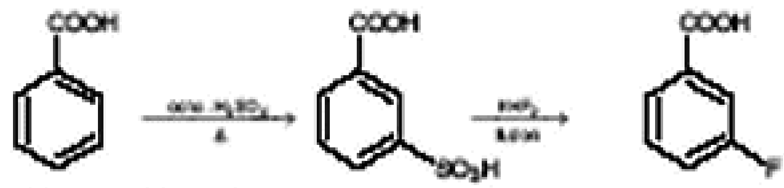
$E_2 = E^0_{cell} - \frac{0.06}{2}\log\frac{C_2}{0.5}$

So, $E_2 - E_1 = \frac{0.06}{2}\left[\log\frac{C_2}{C}\times\frac{0.5}{C_1}\right]$ $\Rightarrow$ $0.03 = \frac{0.06}{2}\log\frac{0.5}{C}$ $\Rightarrow$

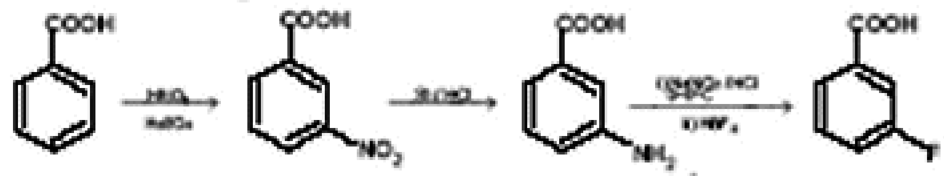
$\log\frac{0.5}{C} = 1$

$C = 0.05\ M$

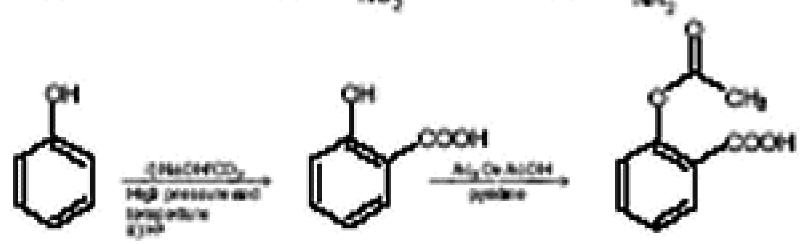
13. a)



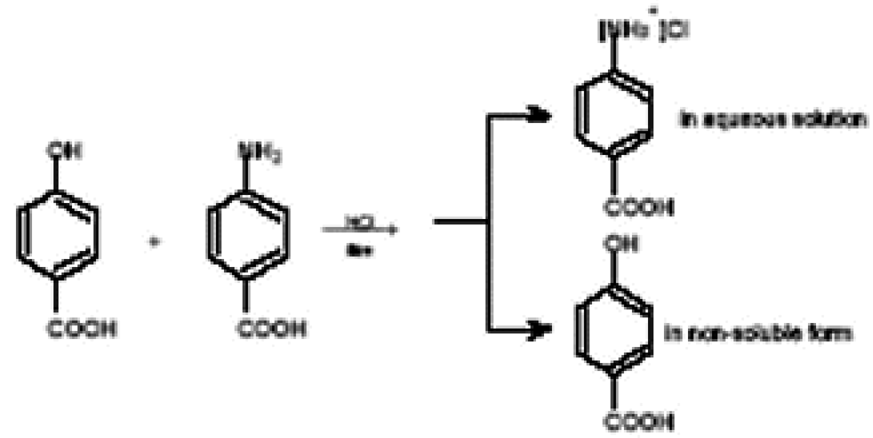
Alternatively

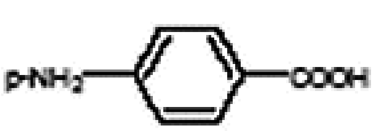
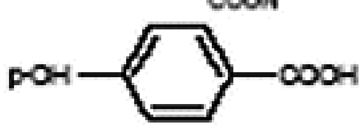


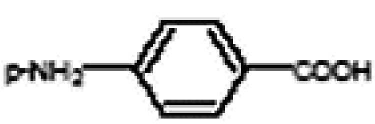
b)

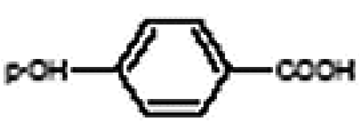


14.

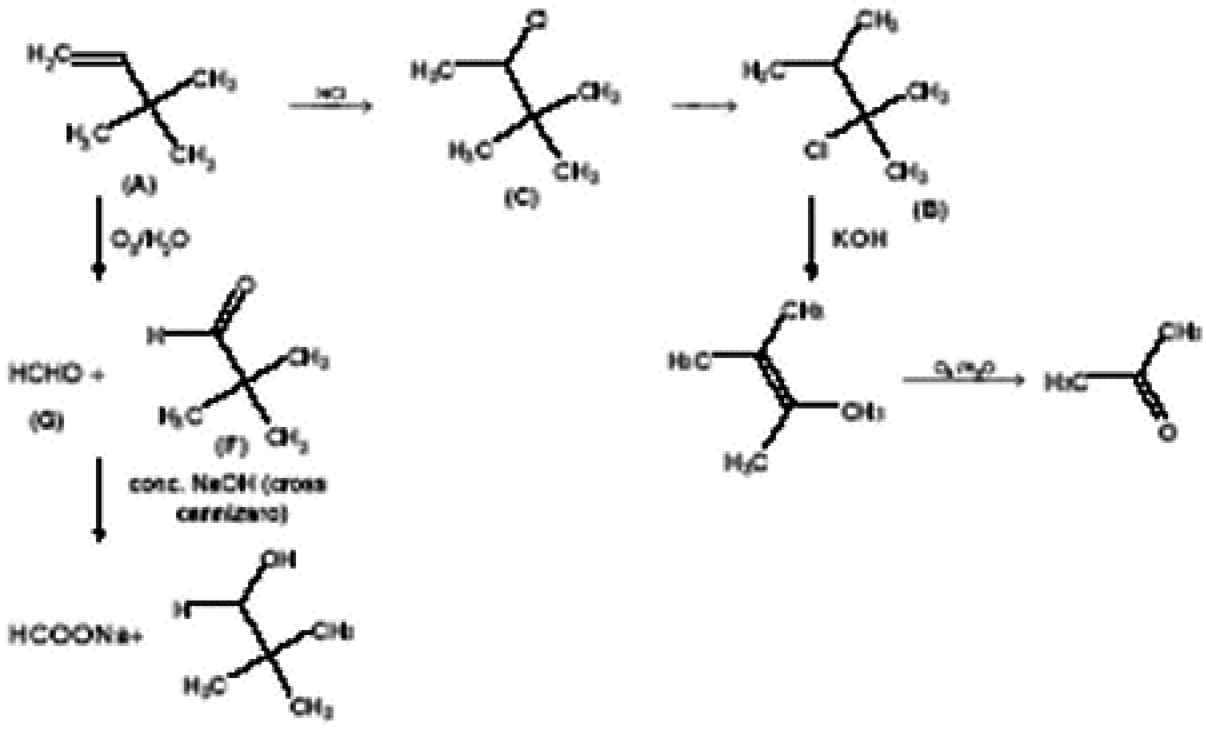


 &  both will give effervesences of CO_2 with $NaHCO_3$

 will give positive azo dye test

 will give positive liebermann nitroso test

15.



16. $Na_2CO_3 + 2SO_2 + H_2O \longrightarrow \underset{(A)}{2NaHSO_3} + CO_2$

$2NaHSO_3 + Na_2CO_3 \longrightarrow \underset{(B)}{2Na_2SO_3} + H_2O + CO_2$

$Na_2SO_3 + S \longrightarrow \underset{(C)}{2Na_2S_2O_3}$

$2Na_2S_2O_3 + I_2 \longrightarrow \underset{(D)}{2Na_2S_4O_6} + 2NaI$

Oxidation states of 'S' are
in (A) and (B) (+4)
in (C) (+6, −2)
in (D) (+5, 0)

17. Potassiumamminetetracyano(C)nitrosoniumchromium(I)
Cr is in + 1 state and d^2sp^3 hybridisation

$\mu = \sqrt{n(n+2)} = \sqrt{3} = 1.73$ B.M.

18. [A = KI, B = HgI_2]

$HgI_2 + H_2S \longrightarrow HgS + 2HI$

$HgI_2 \xrightarrow{\text{aqua-regia}} HgCl_2$

$HgCl_2 + SnCl_2 \longrightarrow \underset{\text{(greyishblack)}}{Hg\downarrow} + SnCl_4$

$2KI + HgI_2 \longrightarrow \underset{\text{orange}}{K_2[HgI_4]}$; $2K_2HgI_4 + NH_3 + 3KOH \longrightarrow \underset{\text{Iodide of Millon's base}}{[HgOHgNH_2]I} + 7KI + 2H_2O$

$Na_2CO_3 + HgI_2 \longrightarrow Hg\downarrow + NaI + CO_2 + O_2\uparrow$

$NaI + Fe^{3+} \longrightarrow \underset{(CCl_4\text{ layer violet})}{I_2 + Na^+} + Fe^{2+}$

$AgNO_3 + I^- \longrightarrow \underset{\substack{\text{Yellow} \\ \text{(insoluble in } NH_3)}}{AgI}$

19. a) $K_b = \frac{RT_b^2}{1000\, l_v} = \frac{RT_b^2 M}{1000 \Delta H_v} = \frac{RT_b M}{1000 \Delta S}$ $\left[\because l_v = \frac{\Delta H_v}{M}\right]$

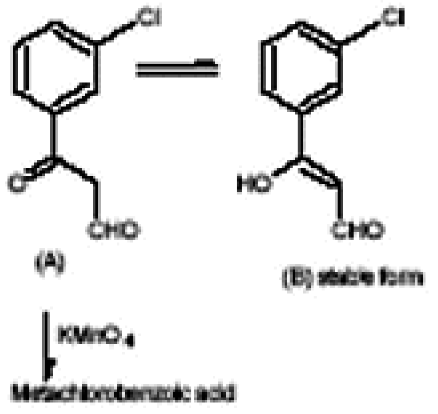
A change from liquid to vapour at boiling point is accompanied by increase in disorderness and hence increase in entropy. However, since a vapour is highly disordered state the difference of the extent of disorderness between vapour and liquid is so high that even if the extent of disorderness varies from liquid to liquid, the same may be considered to be almost equal. In otherwords ΔS may be considered to be constant. M is the same and R is constant so, $K_b \propto T_b$. Thus, $K_b(x) = 0.68$, $K_b(y) = 0.53$ and $K_b(z) = 0.98$.

b) Helium being mono−atomic it has only translational degrees of freedom. The contribution of transitional degree of freedom towards C_v being R/2 so $C_v = 3 \times$

R/2 = 3R/2. Hydrogen molecule is diatomic. However, at low temperature rotational and vibrational contribution are also zero so C_v is 3R/2. At moderate temperature rotational contribution (C = 2× R/2) also becomes dominant and at even higher temperature vibrational contribution (1 × R) also becomes significant.

20. a)

b)



SOLUTIONS

1. (n + 1) division of vernier scale = n division of main scale

$\therefore$ one Vernier division = $\frac{n}{n+1}$ main scale division

Least count = 1 M.S.D. – 1 V.D. = $\frac{1}{n+1}$ M.S. D. = $\frac{a}{n+1}$

2.

For an object placed at infinity the image after first refraction will be formed at v_1

$$\frac{\mu_2}{v_1} - \frac{\mu_1}{-\infty} = \frac{\mu_2 - \mu_1}{+R} \quad \text{... (i)}$$

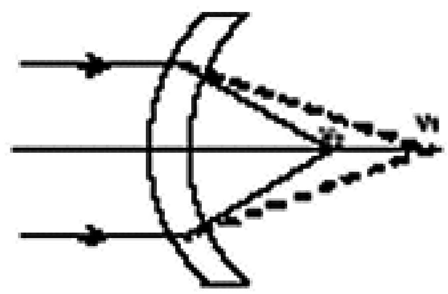
The image after second refraction will be found at v_2

$$\frac{\mu_3}{v_2} - \frac{\mu_2}{v_1} = \frac{\mu_3 - \mu_2}{+R} \quad \text{... (ii)}$$

adding (i) and (ii)

$$\frac{\mu_3}{v_2} = \frac{\mu_3 - \mu_1}{R} \Rightarrow v_2 = \frac{\mu_3 R}{\mu_3 - \mu_1}$$

Therefore focal length will be $\frac{\mu_3 R}{\mu_3 - \mu_1}$



3. $(Z-1)^2 Rhc \left[\frac{1}{1} - \frac{1}{4}\right] = h\nu$

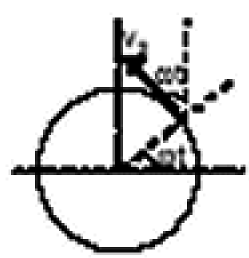
$(Z-1)^2 = \frac{\nu 4}{3RC} \qquad \Rightarrow Z = 42$

4. Loss in K.E. of the gas $\Delta E = \frac{1}{2}(nm)v_0^2$,
where n = number of moles.
If its temperature change by ΔT.

Then $n\frac{3}{2}R\Delta T = \frac{1}{2}(nm)v_0^2$

$\Rightarrow \Delta T = \frac{mv_0^2}{3R}$.

5.



$\omega = \frac{v_2}{R}$

$\vec{v}_2 = (-v_2 \sin\omega t\hat{i} + v_2 \cos\omega t\hat{j})$

$\vec{v}_1 = v_1\hat{j}$

$\vec{v}_{RM} = \vec{v}_2 - \vec{v}_1 = -v_2 \sin\omega t\hat{i} + (v_2 \cos\omega t - v_1)\hat{j}$

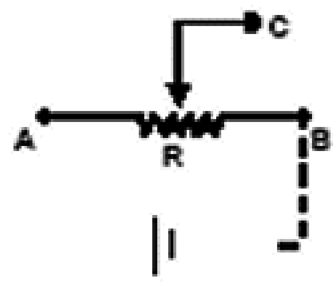
$\vec{P}_{RM} = m\,\vec{v}_{RM} = -mv_2 \sin\omega t\,\hat{i} + m(v_2 \cos\omega t - v_1)\hat{j}$.

6. $(\ell + 0.6r) = \frac{\lambda}{4} = \frac{v}{4f}$

$v = 4f(\ell + 0.6\,r) = 336$ m/s.

7.

Battery should be connected across A and B. Out put can be taken across the terminals A and C or B and C.

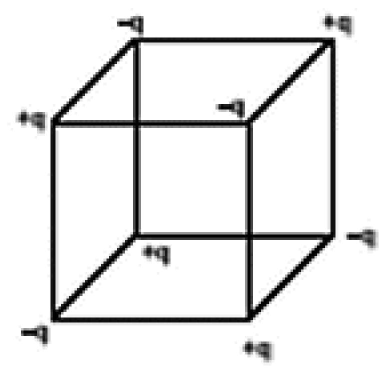


8.

$W_{external} = \Delta PE =$

$$\frac{1}{4\pi\varepsilon_0}\frac{q^2}{a}\left[-\frac{3}{1}+\frac{3}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right]\times\frac{8}{2}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a}\cdot\frac{4}{\sqrt{6}}[3\sqrt{3}-3\sqrt{6}-\sqrt{2}]$$



9.

Let N be the number of active nuclei at time t = 0.

Hence $n = N_0(1 - e^{-2\lambda})$

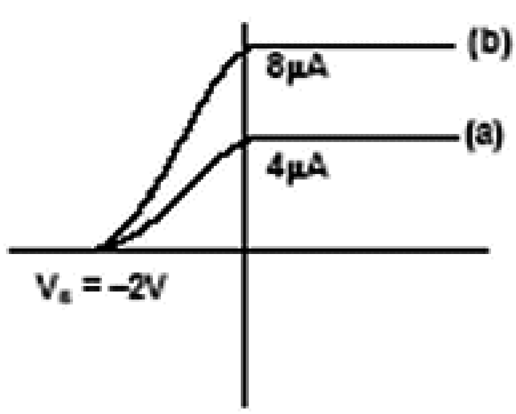
$1.75\,n = N_0(1 - e^{-4\lambda})$

$\Rightarrow \frac{1}{1.75} = \frac{1-e^{-2\lambda}}{1-e^{-4\lambda}}$

$\Rightarrow e^{-4\lambda} - 1.75\,e^{-2\lambda} + 0.75 = 0$

$\Rightarrow \frac{1}{\lambda} = \frac{2}{\ln(4/3)}$ sec.

10.



11. (a) $\frac{dQ}{dt} = \sigma \varepsilon A[(T_\ell)^4 - (T_a)^4]$.

Rate of heat loss per unit area = 595 watt / m^2.

(b) Let T_0 be the temperature of the hot oil

$\Rightarrow \frac{KA(T_0 - T_\ell)}{t} = 595\,A$

$\Rightarrow T_0 \approx 420$ K

12.

(a) $x_1 = v_0 t - A(1 - \cos\omega t)$

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = v_0t$$

$$\Rightarrow x_2 = v_0t + \frac{m_1}{m_2}A(1-\cos\omega t)$$

(b) $$a_1 = \frac{d^2x_1}{dt^2} = -A\omega^2\cos\omega t$$

The separation $(x_2 - x_1)$ between the two blocks will be equal to ℓ_0 when the acceleration will be equal to zero.

$$x_2 - x_1 = \frac{m_1}{m_2}A(1-\cos\omega t) + A(1-\cos\omega t)$$

for $a_1 = 0$

$$x_2 - x_1 = \ell_0 = \left(\frac{m_1}{m_2} + 1\right)A$$

Alternate (b)

In center of mass reference frame, maximum separation of the blocks = $2\ell_0$ (using conservation of energy). If x_1 and x_2 be the separation of the blocks from center of mass at the moment of maximum separation

$x_1 + x_2 = 2\ell_0$ and $m_1x_1 = m_2x_2$

$$\Rightarrow x_1 = 2\ell_0 \Big/ \left(1 + \frac{m_1}{m_2}\right)$$ but $x_1 = 2A$

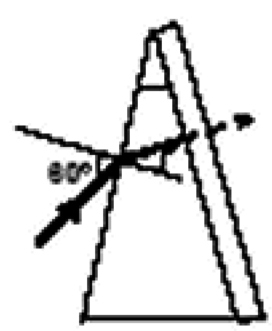
$$\Rightarrow \ell_0 = A\left(1 + \frac{m_1}{m_2}\right)$$

13.

(a) $\mu_{air}\sin 60° = \mu_p \sin r$

$$\Rightarrow \frac{\sqrt{3}}{2} = \sqrt{3}\sin r$$

$\Rightarrow r = 30°$



The refracted ray inside the prism hits the other face at 90°; hence deviation produced by this face is zero and hence angle of emergence is zero.

(b) Multiple reflection occurs between the surfaces of the film

for minimum thickness

$\Delta x = 2\mu t = \lambda$, where t = thickness

$$\Rightarrow t = \frac{\lambda}{2\mu} = 125 \text{ nm}$$

14. For escape velocity from the surface of moon

$$\Rightarrow v_e = \sqrt{\frac{2GM}{R}}$$ where M is the mass of the moon.

P.E. inside the crater of moon will be $= -\frac{GMm}{R} + \int_R^{99R/100} \frac{GMmx}{R^3}dx$

$$= -\frac{199\,GMm}{20000R}$$

K.E. of the body at the highest point will be zero

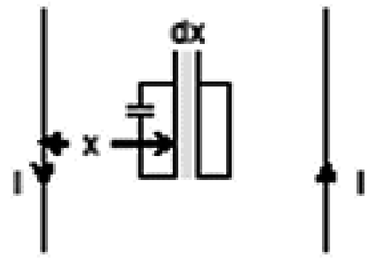
$$-\frac{199GMm}{20000R} + \frac{1}{2}mv_e^2 = -\frac{GMm}{R+h} + 0$$

$\Rightarrow h \approx 100R$

15.

(a) Flux through the square loop

$$= \int_a^{2a} \frac{\mu_0}{4\pi} 2I\left[\frac{1}{x} + \frac{1}{3a-x}\right] a\,dx = \frac{\mu_0}{4\pi} Ia4\ln 2$$



Induced emf $e = -\frac{d\phi}{dt} = -\frac{\mu_0}{\pi} aI_0\omega \ln 2 \cos\omega t$

Charge on the capacitor

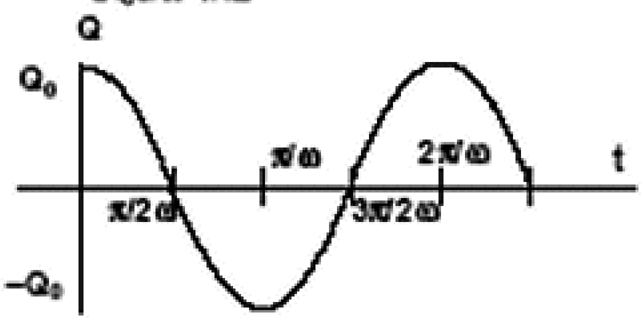
$$Q = Ce = -C\frac{\mu_0}{\pi} aI_0\omega \ln 2 \cos\omega t = -Q_0\cos\omega t \text{ (say)}$$

Current in the loop $= \frac{dQ}{dt}$

$$= \frac{\mu_0}{\pi} CI_0\omega^2 a \ln 2 \sin\omega t$$

$$I_{max} = \frac{\mu_0}{\pi} CI_0 a\omega^2 \ln 2$$

(b)



16. (a) $K.E._{final} = -\Delta P.E. = \vec{P}.\vec{E} = \frac{P}{4\pi\varepsilon_0}\frac{Q^2}{d^2}$

(b) $F = Q\vec{E}$

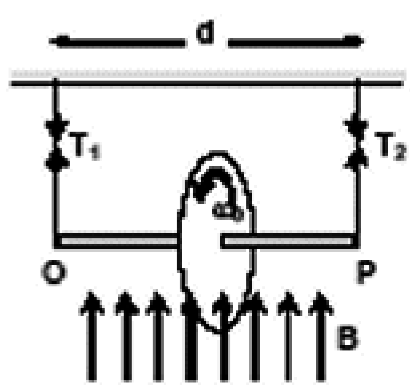
$$= \frac{QP}{2\pi\varepsilon_0 d^3} \text{ along positive x-axis.}$$

17.

$2T_0 = mg$ . . . (1)

For moment of forces about P

$$q\frac{\omega_{max}}{2\pi} \times \pi R^2 \times B + mg\frac{d}{2} = \left(\frac{3T_0}{2}\right)d$$



$$\Rightarrow \omega_{max} = \frac{T_0 d}{qBR^2}$$

18.

Equation of the standing wave in the string is $y = A \sin kx \cos \omega t$

where $k = \pi/\ell$ and $\omega = \frac{\pi}{\ell}\sqrt{\frac{T}{\mu}}$

$\frac{dy}{dt} = -A\omega \sin kx \sin \omega t \Rightarrow v_{max}(x) = A\omega \sin kx$

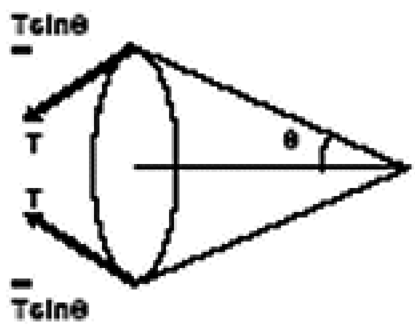
$E = \int_0^{\ell} \frac{1}{2}\mu dx A^2\omega^2 \sin^2 kx = \frac{A^2\pi^2 T}{4\ell}$.

19.

$2\pi b \times 2T\sin\theta = \rho A v^2$

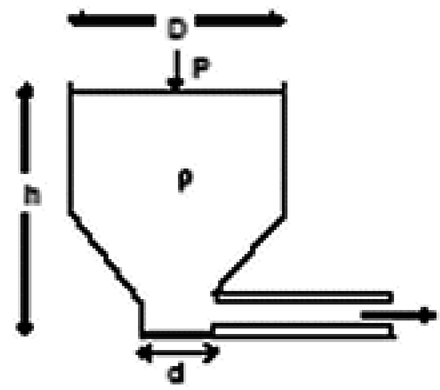
$\Rightarrow 4\pi bT \times \frac{b}{R} = \rho\pi b^2 v^2$

$\Rightarrow R = \frac{4T}{\rho v^2}$



20. When the tube is not there.

$P + h\rho g + \frac{1}{2}\rho v_1^2 = \frac{1}{2}\rho v_0^2$



$\frac{\pi D^2}{4} v_1 = \frac{\pi b^2}{4} v_0$

By Poiseuille's equation the rate of flow of liquid in the capillary tube

$Q = \frac{\pi \Delta P a^4}{8\eta\ell}$

where $\Delta P = P + \rho gh = \frac{1}{2}\rho v_0^2\left[1 - \frac{b^2}{D^2}\right]$

$Q = \frac{1}{2}\rho v_0^2\left[1 - \frac{b^2}{D^2}\right]\frac{\pi a^4}{8\eta\ell}$

$\eta = \frac{\pi a^4 \rho v_0^2}{16 Q\ell}\left[1 - \frac{b^2}{D^2}\right]$

SOLUTIONS

1. To prove $|1 - z_1\bar{z}_2| < |z_1 - z_2|$

$\Leftrightarrow (1 - z_1\bar{z}_2)(1 - \bar{z}_1 z_2) < (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$\Leftrightarrow 1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + |z_1|^2|z_2|^2 < |z_1|^2 - z_2\bar{z}_1 - z_1\bar{z}_2 + |z_2|^2$

$\Leftrightarrow (1 - |z_1|^2) - |z_2|^2(1 - |z_1|^2) < 0$

$\Leftrightarrow (1 - |z_2|^2)(1 - |z_1|^2) < 0$

Which is obvious as $|z_1| < 1 < |z_2|$.

2. $P(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta)$

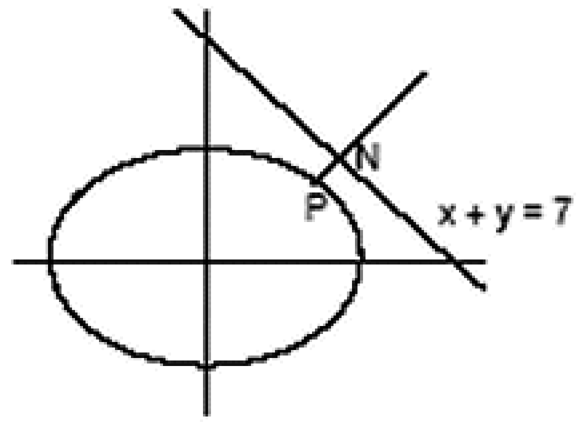
shortest distance exists along the common normal

Slope of normal at P

$= \frac{\sqrt{6}\sec\theta}{\sqrt{3}\operatorname{cosec}\theta} = \sqrt{2}\tan\theta = 1$

so $\cos\theta = \sqrt{\frac{2}{3}}$ and $\sin\theta = \frac{1}{\sqrt{3}}$

Hence $P \equiv (2, 1)$.



3. $A^T A = I$

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow a^2 + b^2 + c^2 = 1$ ....(1)

and $ab + bc + ca = 0$ ....(2)

Now $a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) + 3abc$

$= (a + b + c) + 3$ ...(3)

Now $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$= 1 + 2.0 = 1$

$\Rightarrow a + b + c = 1$ (since a, b, c are real positive number)

Now from (3)

$a^3 + b^3 + c^3 = 1 + 3 = 4$.

Alternate:

$A^T A = I \Rightarrow |A^T A| = |I| \Rightarrow |A|^2 = 1$

$\Rightarrow (a^3 + b^3 + c^3 - 3abc)^2 = 1$

$\Rightarrow a^3 + b^3 + c^3 - 3abc = 1$ (since a, b, c are positive real number $\Rightarrow a^3 + b^3 + c^3 \geq 3abc$ from AM $\geq$ GM)

$\Rightarrow a^3 + b^3 + c^3 = 4$

4. $$\sum_{r=0}^{k}(-1)^r 2^{k-r}\, {}^nC_r\, {}^{n-r}C_{k-r} = \sum_{r=0}^{k}(-1)^r 2^{k-r} \frac{n!}{(n-r)!\,r!}\frac{(n-r)!}{(n-k)!\,(k-r)!}$$

$$= \sum_{r=0}^{k}(-1)^r 2^{k-r} \frac{n!}{(n-k)!\,k!}\frac{k!}{r!\,(k-r)!}$$

$$= {}^nC_k 2^k \sum_{r=0}^{k}\left(-\frac{1}{2}\right)^r {}^kC_r = {}^nC_k 2^k\left(1 - \frac{1}{2}\right)^k = {}^nC_k$$

5. $\int_0^{\pi/2} f(\cos 2x)\cos x\,dx = \int_0^{\pi/4}\left[f(\cos 2x)\cos x + f\left(\cos 2\left(\frac{\pi}{2}-x\right)\right)\cos\left(\frac{\pi}{2}-x\right)\right]dx$

$= \int_0^{\pi/4}\left[f(\cos 2x)\cos x + f(-\cos 2x)\sin x\right]dx$

$= \int_0^{\pi/4} f(\cos 2x)\left[\cos x + \sin x\right]dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos\left(\frac{\pi}{4}-x\right)dx$

$= \sqrt{2}\int_0^{\pi/4} f\left(\cos 2\left(\frac{\pi}{4}-x\right)\right)\cos\left(\frac{\pi}{4}-\left(\frac{\pi}{4}-x\right)\right)dx$

$= \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\,dx$

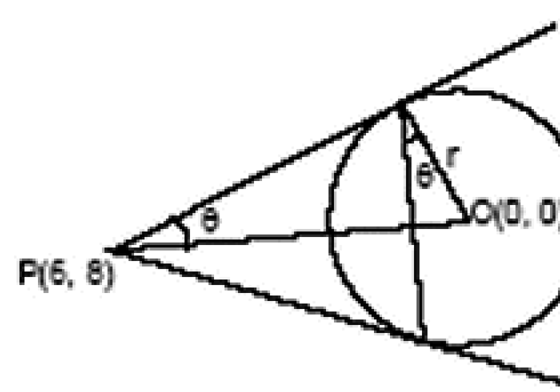
6. Let E_i : denotes the event that the student will pass the ith exam, i = 1, 2, 3
E: denotes the event that the student will qualify.

$P(E) = P(E_1) \times P\left(\frac{E_2}{E_1}\right) + P(E_1) \times P\left(\frac{E'_2}{E_1}\right) \times P\left(\frac{E_3}{E'_2}\right) + P(E'_1) \times P\left(\frac{E_2}{E'_1}\right) \times P\left(\frac{E_3}{E_2}\right)$

$= p^2 + p \times (1-p)\frac{p}{2} + (1-p) \times \frac{p}{2} \times p$

$\Rightarrow P(E) = \frac{2p^2 + p^2 - p^3 + p^2 - p^3}{2} = 2p^2 - p^3$

7. Since OP = 10, $\sin\theta = \frac{r}{10}$ where $\theta \in \left(0, \frac{\pi}{2}\right)$



$A = \frac{1}{2} \times 2r\cos\theta\,(10 - r\sin\theta)$
$= 10\sin\theta\cos\theta\,(10 - 10\sin^2\theta)$
$\Rightarrow A = 100\cos^2\theta\sin\theta\cos\theta$

$\frac{dA}{d\theta} = 100\left[\cos^4\theta - 3\sin^2\theta\cos^2\theta\right]$

$= 300\cos^4\theta\left(\frac{1}{\sqrt{3}} - \tan\theta\right)\left(\frac{1}{\sqrt{3}} + \tan\theta\right)$

$\Rightarrow$ A is maximum at $\theta = \frac{\pi}{6} \Rightarrow r = 10 \times \frac{1}{2} = 5$ units.

8 $a_1z + a_2z^2 + \ldots. + a_nz^n = 1$
$\Rightarrow |a_1z + a_2z^2 + \ldots. + a_nz^n| = 1$
$\Rightarrow |a_1z| + |a_2z^2| + \ldots. + |a_nz^n| \geq 1$
$\Rightarrow 2[|z| + |z|^2 + \ldots. + |z|^n] > 1$

$\Rightarrow \frac{2|z|\left(1-|z|^n\right)}{(1-|z|)} > 1$

as $|z| < \frac{1}{3}$, $|z| > \frac{1}{3} + \frac{2}{3}|z|^{n+1} \Rightarrow |z| > \frac{1}{3}$
This is a contradiction.
Hence there exists no such complex number.

9. P(A) = Probability that A will hit B
P(B) = Probability that B will hit A
P(C) = Probability that C will hit A

P(E) = Probability that A will be hit

Then P(E) = $1-P(\overline{B}\cap\overline{C}) = 1-P(\overline{B}).P(\overline{C}) = 1-\frac{1}{2}.\frac{2}{3} = \frac{2}{3}$

$P\left(\frac{B\cap\overline{C}}{E}\right) = \frac{P(B).P(\overline{C})}{P(E)} = \frac{\frac{1}{2}.\frac{2}{3}}{\frac{2}{3}} = \frac{1}{2}$

10. L.H.D. at x = a

$f'_-(a) = \lim_{h\to0^-}\frac{f(a+h)-f(a)}{h} = 0$ (given)

$f'_-(-a) = \lim_{h\to0^-}\frac{f(-a+h)-f(-a)}{h} = \lim_{h\to0^-}\frac{-f(-h+a)+f(a)}{h}$ (As f is odd)

$= \lim_{h\to0^-}\frac{-f(2a+h-a)+f(a)}{h} = -f'_-(a) = 0.$

11. Let f (x) = sin (tan x) – x

$f'(x) = \cos(\tan x)\sec^2 x - 1 = \tan^2 x\cos(\tan x) + \cos(\tan x) - 1 > \tan^2 x\cos(\tan x) - \frac{\tan^2 x}{2}$

$\Rightarrow f'(x) > \tan^2 x(\cos(\tan x) - \cos\frac{\pi}{3}) > 0$ (since $0 \le \tan x \le 1 < \frac{\pi}{3}$)

$\Rightarrow$ f(x) is an increasing function $\forall\ x \in \left[0, \frac{\pi}{4}\right]$

As f(0) = 0 $\Rightarrow f(x) \ge 0\ \forall\ x \in \left[0, \frac{\pi}{4}\right]$

$\Rightarrow \sin(\tan x) \ge x.$

12. 2b = a + c ...(1)

$b^2 = \frac{2a^2c^2}{a^2+c^2} = \left(\frac{a+c}{2}\right)^2$

$\Rightarrow (a^2 + c^2)^2 + 2ac(a^2 + c^2) = 8a^2c^2$

$\Rightarrow (a^2 + c^2 + ac)^2 = 9a^2c^2$

$\Rightarrow a^2 + c^2 + ac = \pm 3ac$ ...(2)

$\Rightarrow a^2 + c^2 - 2ac = 0 \Rightarrow a = c \Rightarrow b = c$

or, $a^2 + c^2 = -4ac$

$\Rightarrow (a + c)^2 = -2ac$

$\Rightarrow 4b^2 = -2ac \Rightarrow b^2 = -\frac{ac}{2}$

$\Rightarrow a, b, -\frac{c}{2}$ are in G.P.

13. For unequal real roots

D > 0

$\Rightarrow (a-b)^2 - 4(1-a-b) > 0$

$\Rightarrow b^2 + b(4-2a) + a^2 + 4a - 4 > 0$

For the above quadratic expression to be true $\forall\ b \in R$

Discriminant of its corresponding equation should be less than zero

i.e. $(4-2a)^2 - 4(a^2 + 4a - 4) < 0$

$\Rightarrow -32a + 32 < 0$

$\Rightarrow a > 1$.

14. Let the point P be (h, k)

$\Rightarrow k = mh - 2m - m^3$ or, $m^3 + m(2-h) + k = 0$

$\Rightarrow m_1m_2m_3 = -k \Rightarrow m_3 = -\frac{k}{\alpha}$

$\Rightarrow \left(-\frac{k}{\alpha}\right)^3 - \frac{k}{\alpha}(2-h) + k = 0$

$\Rightarrow k^2 = \alpha^2 h - 2\alpha^2 + \alpha^3$

$\Rightarrow y^2 = \alpha^2 x - 2\alpha^2 + \alpha^3$

Comparing it with $y^2 = 4x$, we get $\alpha^2 = 4$ and $-2\alpha^2 + \alpha^3 = 0 \Rightarrow \alpha = 2$.

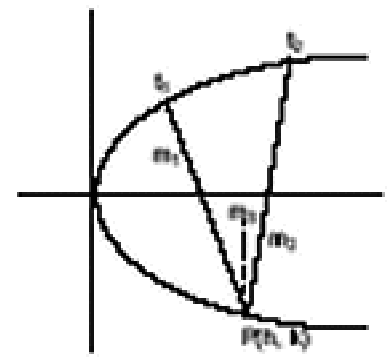
Alternate:

Since locus of P is a part of the parabola $\Rightarrow$ normals at any two points t_1 and t_2 meet at P

$\Rightarrow t_1t_2 = 2$

$\Rightarrow (-m_1)(-m_2) = 2$

$\Rightarrow \alpha = 2$



15(i). From Lagrange's mean value theorem

$\frac{f(4)-f(0)}{4-0} = f'(a)$ for $a \in (0, 4)$ ...(1)

Also from Intermediate mean value theorem

$\frac{f(4)+f(0)}{2} = f(b)$ for $b \in (0, 4)$ ...(2)

From (1) and (2), we get

$\frac{(f(4))^2-(f(0))^2}{8} = f'(a)f(b)$. for $a, b \in (0, 4)$

(ii). Replacing t by z^2, we get $\int_0^4 f(t)dt = \int_0^2 2zf(z^2)\,dz$

From Lagrange's mean value theorem

$\frac{\int_0^2 2zf(z^2)dz - \int_0^0 2zf(z^2)dz}{2-0} = 2\gamma f(\gamma^2)$ for $\gamma \in (0, 2)$

$\Rightarrow \int_0^2 2zf(z^2)dz = 2(2\gamma f(\gamma^2)) = 2\left(\frac{2\alpha f(\alpha^2)+2\beta f(\beta^2)}{2}\right)$ (where $0< \alpha < \gamma <\beta < 2$, using intermediate mean value theorem)

$\Rightarrow \int_0^4 f(t)\,dt = 2[\alpha f(\alpha^2)+\beta f(\beta^2)]$ $\forall\ 0 < \alpha, \beta < 2$.

16(i). The equation of the plane is $\begin{vmatrix} x-2 & y-1 & z-0 \\ 5-2 & 0-1 & 1-0 \\ 4-2 & 1-1 & 1-0 \end{vmatrix} = 0 \Rightarrow (x-2)(-1) - (y-1)(1) + z(2) = 0$

$\Rightarrow x + y - 2z = 3$.

(ii). Let Q be (α, β, γ).

Equation of line PQ is $\frac{x-2}{1} = \frac{y-1}{1} = \frac{z-6}{-2}$

$\Rightarrow \frac{\frac{\alpha+2}{2}-2}{1} = \frac{\frac{\beta+1}{2}-1}{1} = \frac{\frac{\gamma+6}{2}-6}{-2}$

$$= \frac{1\left(\frac{\alpha+2}{2}-2\right)+1\left(\frac{\beta+1}{2}-1\right)-2\left(\frac{\gamma+6}{2}-6\right)}{1.1+1.1+(-2)(-2)} = 2 \qquad \left(\text{Since } 1\left(\frac{\alpha+2}{2}\right)+1\left(\frac{\beta+1}{2}\right)-2\left(\frac{\gamma+6}{2}\right)=3\right)$$

$\Rightarrow \alpha = 6,\ \beta = 5,\ \gamma = -2 \Rightarrow Q \equiv (6, 5, -2)$.

17. $\frac{dP(x)}{dx} > P(x) \Rightarrow \frac{dP(x)}{dx} - P(x) > 0$

$\Rightarrow \frac{d}{dx}(P(x)e^{-x}) > 0$

$\Rightarrow P(x) \cdot e^{-x}$ is an increasing function.

$\Rightarrow P(x)\, e^{-x} > P(1)\, e^{-1}\ \forall\, x \ge 1$

$\Rightarrow P(x)\, e^{-x} > 0\ \forall\, x > 1$ (since $P(1) = 0$)

$\Rightarrow P(x) > 0\ \forall\, x > 1$.

18. $I_n = \frac{n}{2} r^2 \sin\frac{2\pi}{n} \Rightarrow \frac{2I_n}{n} = \sin\frac{2\pi}{n}$ ....(1)

$O_n = nr^2 \tan\frac{\pi}{n}$ ....(2)

From (1) and (2), we get

$$\frac{I_n}{O_n} = \frac{1}{2}\frac{\sin\frac{2\pi}{n}}{\tan\frac{\pi}{n}} = \cos^2\frac{\pi}{n} = \frac{\cos\frac{2\pi}{n}+1}{2} = \frac{1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}}{2} \quad (\text{using } (1))$$

$$I_n = \frac{O_n}{2}\left(1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right)$$

19. $\vec{x} = \frac{\vec{u}+\vec{v}}{|\vec{u}+\vec{v}|} = \frac{1}{2}\sec\frac{\alpha}{2}(\vec{u}+\vec{v})$. Similarly for vectors $\vec{y}$ and $\vec{z}$

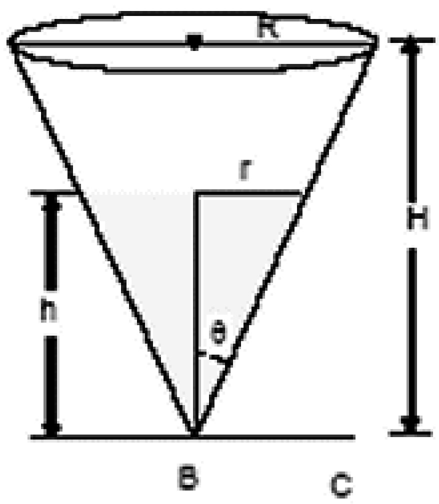
As $[(\vec{x}\times\vec{y})\ \ (\vec{y}\times\vec{z})\ \ (\vec{z}\times\vec{x})] = [\vec{x}\ \vec{y}\ \vec{z}]^2$

$= \frac{1}{64}\sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}[\vec{u}+\vec{v}\ \ \vec{v}+\vec{w}\ \ \vec{w}+\vec{u}]^2$

$= \frac{4}{64}\sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}[\vec{u}\ \ \vec{v}\ \ \vec{w}]^2$ As $[\vec{u}+\vec{v}\ \ \vec{v}+\vec{w}\ \ \vec{w}+\vec{u}] = 2[\vec{u}\ \vec{v}\ \vec{w}]$

$= \frac{1}{16}\sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}[\vec{u}\ \ \vec{v}\ \ \vec{w}]^2$

20. Let the semi vertical angle of the cone be $\theta = \tan^{-1}\left(\frac{R}{H}\right)$



Let height of the liquid at time 't' be 'h' from the base BC and radius r.

Volume of liquid at time 't' = V = $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^3 \cot\theta$

S = Surface area in contact with air at time 't' = πr^2

Given that $-\frac{dV}{dt} \propto S$

$\Rightarrow -\frac{dV}{dt} = kS = k\pi r^2$

$$\Rightarrow \frac{\cot\theta}{3}\pi 3r^2 \frac{dr}{dt} = -k\pi r^2$$

$$\Rightarrow \cot\theta \int_R^0 dr = -k\int_0^T dt$$ (where T is the required time)

$$\Rightarrow \frac{H}{R}R = kT \Rightarrow T = \frac{H}{k}$$

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